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Thermal Performance of a Functionally Graded Radial Fin

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Abstract This paper investigates the steady-state thermal performance of a radial fin of rectangular profile made of a functionally graded material. The thermal conductivity of the fin varies continuously in the radial direction following a power law. The boundary conditions of a constant base temperature and an insulated tip are assumed. Analytical solutions for the temperature distribution, heat transfer rate, fin efficiency, and fin effectiveness are found in terms of Airy wave functions, modified Bessel functions, hyperbolic functions, or power functions depending on the exponent of the power law. Numerical results illustrating the effect of the radial dependence of the thermal conductivity on the performance of the fin are presented and discussed. It is found that the heat transfer rate, the fin efficiency, and the fin effectiveness are highest when the thermal conductivity of the fin varies inversely with the square of the radius. These quantities, however, decrease as the exponent of the power law increases. The results of the exact solutions are compared with a solution derived by using a spatially averaged thermal conductivity. Because large errors can occur in some cases, the use of a spatially averaged thermal-conductivity model is not recommended.

Keywords Fin effectiveness · Fin efficiency · Functionally graded radial fin · Heat transfer rate · Spatially averaged thermal-conductivity model

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List of Symbols

a	Parameter in thermal-conductivity expression
A	Constant
Ai	Airy function of the first kind
Bi	Airy function of the second kind
C_1, C_2	Constants
d	Constant
D	Denominator
f_1, f_2	Constants
F	Function of r_b, r_t, and n
h	Convective heat transfer coefficient
I	Modified Bessel function of first kind
J	Bessel function of the first kind
k	Fin thermal conductivity
K	Modified Bessel function of second kind
m	Fin parameter
n	Exponent
q	Fin heat transfer rate
r	Radial coordinate
T	Temperature
w	Fin thickness
Y	Bessel function of the second kind

Greek Symbols

ε	Fin effectiveness
η	Fin efficiency
θ	Excess temperature

Subscripts

a	Ambient
b	Fin base
t	Fin tip

1 Introduction

Functionally graded materials (FGM) are finding increasing use in rocket heat shields, heat exchanger tubes, thermoelectric generators, heat-engine components, plasma facings for fusion reactors, and electrically insulating metal/ceramic joints. The continuous spatial variation of thermophysical properties, such as the thermal conductivity and the heat capacity, can offer advantages that are not available with the use of homogeneous materials. For example, a thin functionally graded thermal shield can sustain steep temperature gradients without excessive thermal stresses. Similar advantages can be realized with functionally graded heat exchanger tubes and heat engine components. Because radial fins of rectangular profile are often manufactured integrally with the tubes to enhance the heat transfer rate from the tube, it is important to have a knowledge of the heat transfer characteristics of radial fins made of functionally graded materials if the performance of such a finned heat exchanger is to be properly evaluated.

The analysis of heat transfer in a radial fin made of a homogeneous material having a rectangular profile and spatially uniform thermal conductivity is well documented in the literature [1,2]. For a fin made of a functionally graded material, the spatial dependence of thermal conductivity must be taken into account to accurately predict the thermal performance of the fin. According to Babaaei and Chen [3], the vast majority of research studies such as those by Noda [4], Eslami et al. [5], and Hosseini et al. [6] use a Fourier conduction model to investigate heat conduction in heterogeneous materials. Following this research approach, which is adequate for steady-state heat conduction analysis, it is assumed that heat conduction in the functionally graded fin obeys the Fourier law.

Some workers [7–9] have studied radial fins with a two-dimensional heat conduction model and have found that when the Biot number based on the thickness of the fin is of the order of 0.1, the error between the one-dimensional and two-dimensional models is negligible. Indeed, it has been established that the fin is effective as a heat transfer enhancement device only when the Biot number is small [10]. Thus, for a properly designed fin, heat conduction is predominantly in the radial direction.

The thermal conductivity of the fin is assumed to vary with the radial coordinate according to the power law whose exponent is a measure of the nonhomogeneity of the fin material. Such a power-law type of spatial variation of the thermal conductivity has been adopted by Sahin [11] in the study of optimal distribution of insulation on a flat hot surface and by Babaei and Chen [3] in analyzing one-dimensional, transient hyperbolic heat conduction in a functionally graded hollow sphere.

Based on the foregoing discussion, this paper uses a one-dimensional steady-state Fourier heat conduction model to investigate the thermal performance of a functionally graded radial fin of rectangular shape in which the thermal conductivity varies as a power function of the radial coordinate. The standard boundary conditions of a constant base temperature and an adiabatic tip are used to derive exact analytical solutions for the temperature distribution, the heat transfer rate, the efficiency, and the effectiveness of the fin. These solutions appear in terms of the Airy wave functions, the modified Bessel functions, the hyperbolic functions, or the power functions depending on the exponent of the power law variation. Numerical results illustrating the effect of a variable thermal conductivity on the performance of the fin are presented and discussed. The paper also develops a solution based on a spatially averaged thermal-conductivity model and compares its predictions with those of exact solutions.

2 Analysis

Figure 1 shows a radial fin of rectangular profile made of a functionally graded material. The fin has base radius r_b, tip radius r_t, and thickness w. The fin is operating in a convective environment which is characterized by the heat transfer coefficient h and ambient temperature T_a. The radial coordinate r is measured from the centerline

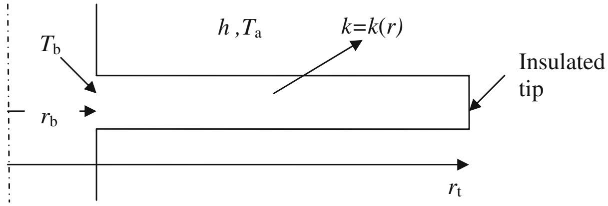

Fig. 1 Radial fin of rectangular profile

(shown *dotted*) of the tube to which the fin is attached. Unlike the standard analysis, which assumes the fin material to be homogeneous with uniform thermal conductivity, the present analysis models the nonhomogeneity of the fin by allowing the fin thermal conductivity k to vary as a power function of radial coordinate r, that is,

$$k = ar^n \tag{1}$$

where n and a are constants. Such a power dependence of the thermal conductivity occurs in some aerospace and automotive structures [11]. The fin is assumed to be at a constant temperature T_b at the base ($r = r_b$) and its tip ($r = r_t$) to be adiabatic.

With the use of Eq. 1, the one-dimensional steady-state heat conduction equation for the fin together with the boundary conditions may be written as

$$\frac{d^2\theta}{dr^2} + (n+1)\frac{1}{r}\frac{d\theta}{dr} - m^2 r^{-n}\theta = 0, \tag{2}$$

$$r = r_b, \quad \theta = \theta_b, \tag{3a}$$

$$r = r_t, \quad \frac{d\theta}{dr} = 0 \tag{3b}$$

where $\theta = T - T_a$ and $m^2 = 2h/aw$.

The fin heat transfer rate q may be found as

$$q = -(ar\,b^n)2\pi r_b w \left.\frac{d\theta}{dr}\right|_{r=r_b}. \tag{4}$$

The fin efficiency η defined as the ratio of actual heat transfer rate and the ideal heat transfer rate for a fin of infinite thermal conductivity is given by

$$\eta = q/q_{\text{ideal}} = q/h2\pi\left(r_t^2 - r_b^2\right)\theta_b. \tag{5}$$

Finally, the fin effectiveness ε defined as the ratio of actual heat transfer and the heat transfer without fin may be obtained from

$$\varepsilon = q/2\pi r_b w h\theta_b \tag{6}$$

3 Exact Solutions

The exact analytical solution of Eq. 2 is found to be

$$\theta = r^{-n/2}\left[C_1 J_p\left(i\frac{2m}{2-n}r^{2-n/2}\right) + C_2 Y_p\left(i\frac{2m}{2-n}r^{2-n/2}\right)\right] \tag{7}$$

where $p = \text{abs}[n/2 - n]$ and C_1 and C_2 are constants of integrations which may be found by applying the boundary conditions of Eqs. 3a and 3b. The symbols J and Y represent the Bessel functions of the first and second kind, respectively. We study the cases of $n = -1$, -2, 1, and 2 as a representative sample. The case of $n = 0$ (constant thermal conductivity) is well known but would be included here for completeness.

Case 1: $n = -1$.

In this case, the solution of Eq. 2 subject to the boundary conditions, Eqs. 3a and 3b, is found to be

$$\frac{\theta}{\theta_b} = \frac{Ai'(m^{2/3}r_t)Bi(m^{2/3}r) - Bi'(m^{2/3}r_t)Ai(m^{2/3}r)}{Ai'(m^{2/3}r_t)Bi(m^{2/3}r_b) - Ai(m^{2/3}r_b)Bi'(m^{2/3}r_t)} \tag{8}$$

where Ai and Bi are the Airy functions of the first and second kind, respectively, and the primes denote the derivatives. Tables of Airy functions and their derivatives are provided by Abramowitz and Stegun [12] . The calculation of Airy functions is greatly facilitated by the use of the popular symbolic algebra packages such as Maple and Mathematica which provide a library of many built-in special functions [13].

The solution for q is found to be

$$q = 2\pi a w\theta_b m^{2/3}\left[\frac{Ai'(m^{2/3}r_b)Bi'(m^{2/3}r_t) - Ai'(m^{2/3}r_t)Bi'(m^{2/3}r_b)}{Ai'(m^{2/3}r_t)Bi(m^{2/3}r_b) - Ai(m^{2/3}r_b)Bi'(m^{2/3}r_t)}\right]. \tag{9}$$

Once q has been obtained from Eq. 8, the fin efficiency η and fin effectiveness ε may be calculated from Eqs. 5 and 6, respectively.

Case 2: $n = -2$.

The solutions for θ and q are given by

$$\begin{aligned}\theta/\theta_b = &\left\{(\cosh f_1 + \sinh f_1)\left\{\left[(\cosh f_2 + \sinh f_2)^2 + 1\right]\cosh\left(\frac{1}{2}mr^2\right)\right.\right.\\ &\left.\left.-\left[(\cosh f_2 + \sinh f_2)^2 - 1\right]\sinh\left(\frac{1}{2}mr^2\right)\right\}\right\} \Big/ \left\{(\cosh f_1 + \sinh f_1)^2\right.\\ &\left.+(\cosh f_2 + \sinh f_2)^2\right\}, \qquad (10)\\ q = &\ 2\pi a w m\theta_b \frac{\sinh f_2 \cosh f_1 - \cosh f_2 \sinh f_1}{\cosh f_1 \cosh f_2 - \sinh f_1 \sinh f_2}, \qquad (11)\end{aligned}$$

with $f_1 = (1/2)mr_b^2$ and $f_2 = (1/2)mr_t^2$.

Case 3: $n = 1$.

For $n = 1$, the solutions for θ and q are given by

$$\frac{\theta}{\theta_\mathrm{b}} = \frac{\sqrt{r_\mathrm{b}}\left[I_1(2m\sqrt{r})K_1(2c_2) + c_2 I_1(2m\sqrt{r})K_0(2c_2) - K_1(2m\sqrt{r})I_1(2c_2) + c_1 K_1(2m\sqrt{r})I_0(2c_2)\right]}{\sqrt{r}\left[-I_1(2c_2)K_1(2c_1) + c_2 I_0(2c_2)K_1(2c_1) + K_1(2c_2)I_1(2c_1) + c_2 K_0(2c_2)I_1(2c_1)\right]}, \tag{12}$$

$$q = -2\pi a w \theta_\mathrm{b} r_\mathrm{b} \frac{N}{D}, \tag{13}$$

where

$$\begin{aligned} N &= I_1(2c_2)K_1(2c_1) - K_1(2c_2)I_1(2c_1) - c_2 K_0(2c_2)I_1(2c_1) \\ &\quad + c_1 K_1(2c_2)I_0(2c_1) - c_2 I_0(2c_2)K_1(2c_1) + c_1 c_2\left[K_0(2c_2)I_0(2c_1)\right. \\ &\quad \left. + I_0(2c_2)K_0(2c_1)\right], \\ D &= c_2\left[I_0(2c_2)K_1(2c_2) + K_0(2c_2)I(2c_1)\right] + K_1(2c_2)I_1(2c_1) - I_1(2c_2)K_1(2c_1), \\ c_1 &= m\sqrt{r_\mathrm{b}},\ c_2 = m\sqrt{r_\mathrm{t}}, \end{aligned}$$

and I, K are modified Bessel functions of first and second kinds, respectively, and with subscripts denoting the order of the function.

Case 4: $n = 2$.

For $n = 2$, the solutions for θ and q are given by

$$\frac{\theta}{\theta_\mathrm{b}} = \left[\frac{(1+d)r^{2d} - (1-d)r_\mathrm{t}^{2d}}{(1+d)r_\mathrm{b}^{2d} - (1-d)r_\mathrm{t}^{2d}}\right]\left[\frac{r_\mathrm{b}}{r}\right]^{1+d}, \tag{14}$$

$$q = 2\pi a w m^2 r_\mathrm{b}^2 \theta_\mathrm{b}\left[\frac{r_\mathrm{t}^{2d} - r_\mathrm{b}^{2d}}{(1+d)r_\mathrm{b}^{2d} - (1-d)r_\mathrm{t}^{2d}}\right], \tag{15}$$

where $d = (1 + m^2)^{1/2}$.

Case 5: $n = 0$ (constant thermal conductivity).

For $n = 0$ which corresponds to a constant thermal conductivity, the solutions for θ and q are as follows:

$$\frac{\theta}{\theta_\mathrm{b}} = \frac{I_0(mr)K_1(mr_\mathrm{t}) + K_0(mr)I_1(mr_\mathrm{t})}{I_0(mr_\mathrm{b})K_1(mr_\mathrm{t}) + K_0(mr_\mathrm{b})I_1(mr_\mathrm{t})}, \tag{16}$$

$$q = 2\pi a w m r_\mathrm{b}\theta_\mathrm{b}\frac{K_1(mr_\mathrm{b})I_1(mr_\mathrm{t}) - I_1(mr_\mathrm{b})K_1(mr_\mathrm{t})}{K_0(mr_\mathrm{b})I_1(mr_\mathrm{t}) + I_0(mr_\mathrm{b})K_1(mr_\mathrm{t})}. \tag{17}$$

4 Numerical Results

The exact solutions presented in the foregoing section show that the excess temperature θ and the fin heat transfer rate q depend on the base radius r_b, the tip radius r_t, the base temperature excess θ_b, the fin thickness w, the thermal conductivity parameter a, and the conventional fin parameter m. Because of six variables, the results are presented

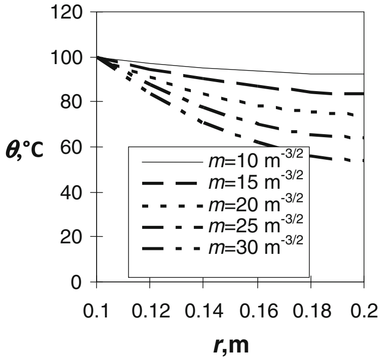

Fig. 2 Temperature distributions in the fin for Case 1

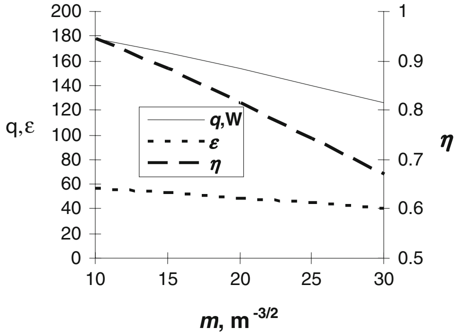

Fig. 3 Variations of heat transfer rate, fin effectiveness, and fin efficiency with fin parameter m for Case 1

for a specific annular fin of base radius $r_b = 0.01$ m, tip radius $r_t = 0.02$ m, thickness $w = 0.005$ m and with a base excess temperature of 100 °C. A convection heat transfer coefficient of $10\,\mathrm{W \cdot m^{-2} \cdot K^{-1}}$ is assumed to exist on the exposed surfaces of the fin.

It should be pointed out that since the thermal conductivity of the fin varies with the distance r, the transverse Biot number given by $Bi = hw/ar^n$ is not a constant for all radial locations. In particular, for negative values of n and large values of r, the Biot number may exceed 0.1 which is traditionally used as the limit for the applicability of the one-dimensional heat model conduction used in this work [14].

The results for the five cases are now discussed.

Case 1: $n = -1$.

The value of the thermal conductivity parameter a was varied to give $m = (10, 15, 20, 25, \text{and } 30)\,\mathrm{m^{-3/2}}$. The temperature distributions for these values of m are shown in Fig. 2. As a decreases, that is, the thermal conductivity of the fin decreases, and thus, m increases (h and w are fixed), the temperature distribution in the fin becomes progressively steeper. This behavior is synonymous with that of a constant thermal conductivity fin. The fin heat transfer rate, the fin effectiveness, and the fin efficiency as functions of m are illustrated in Fig. 3. As expected, with the increase in m (decrease in thermal conductivity), all three quantities decrease. Because of the low value of h ($10\,\mathrm{W \cdot m^{-2} \cdot K^{-1}}$), the fin effectiveness is high. As the thermal conductivity decreases, i.e., m increases, the fin effectiveness decreases.

Cases: 2, 3, 4, and 5.

The results for Cases 2, 3, 4, and 5 exhibit the same characteristics as the results for Case 1 and will be omitted in favor of comparative results for all five cases.

5 Comparative Results

Consider an annular fin of base radius $r_b = 0.01$ m, tip radius $r_t = 0.02$ m, and thickness $w = 0.005$ m operating with a base excess temperature $\theta_b = 100$ °C in a convective environment which provides a heat transfer coefficient of $30\,\mathrm{W \cdot m^{-2} \cdot K^{-1}}$. The thermal conductivity parameter a is chosen as $10\,\mathrm{W \cdot m^{-(n+1)} \cdot K^{-1}}$.

To aid the discussion, Table 1 has been prepared showing the thermal conductivities k_b and k_t at the base and tip, respectively, for the five values of n. The effect of varying the thermal conductivity exponent n on the fin tip temperature excess is shown in Fig. 4. Because the thermal conductivities for $n = -2$ are the highest, the highest tip temperature occurs for $n = -2$. As n increases, the thermal conductivities decrease and consequently the tip temperature also decreases. For $n = 1$ and $n = 2$, the tip is virtually at the temperature of the environment because of the low values of thermal conductivities.

The effect of varying the exponent n on the fin heat transfer rate is depicted in Fig. 5. As n increases, the heat dissipation capacity of the fin diminishes. This diminution is caused by the reduction in thermal-conductivity values. Figure 5 shows that the highest heat dissipation from the fin occurs when its thermal conductivity varies inversely with the square of the radius, i.e., $n = -2$. For this case, the tip temperature (Fig. 4) is about 88 °C, a departure of only 12 °C from the base to the tip. Thus the entire

Table 1 Thermal conductivities ($\mathrm{W \cdot m^{-1} \cdot K^{-1}}$) at the base and tip of the fin

	$n = -2$	$n = -1$	$n = 0$	$n = 1$	$n = 2$
k_b	1000	100	10	1	0.1
k_t	250	50	10	1	0.4

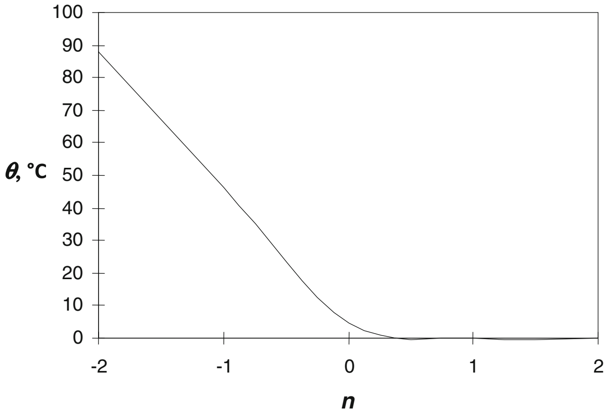

Fig. 4 Variation of fin tip temperature with exponent n with $a = 10\,\mathrm{W \cdot m^{-(n+1)} \cdot K^{-1}}$

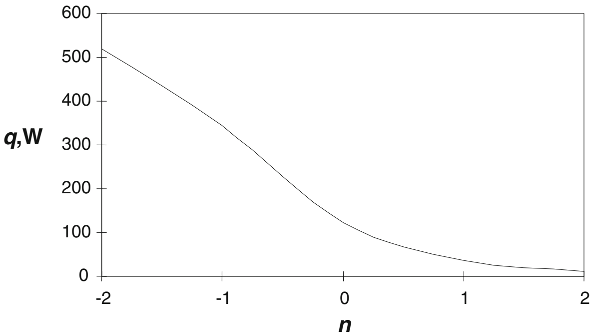

Fig. 5 Heat transfer rate versus the exponent n with $a = 10\,\mathrm{W} \cdot \mathrm{m}^{-(n+1)} \cdot \mathrm{K}^{-1}$

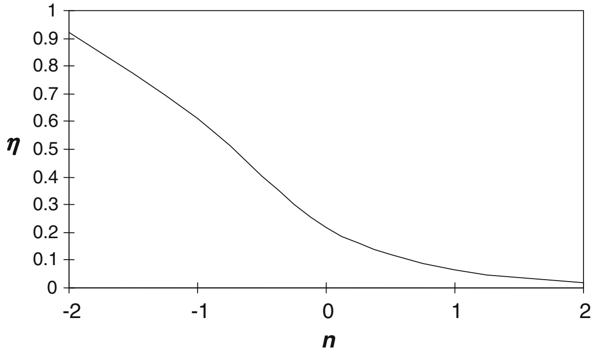

Fig. 6 Fin efficiency as a function of exponent n with $a = 10\,\mathrm{W} \cdot \mathrm{m}^{-(n+1)} \cdot \mathrm{K}^{-1}$

exposed surface (bottom and top) is providing efficient heat dissipation by convection to the surroundings. From a design perspective, the fin material should have the highest attainable thermal conductivity at the base of the fin and the material should be graded so the thermal conductivity progressively decreases toward the fin tip.

Figure 6 is a plot of the fin efficiency as a function of exponent n. The fin efficiency drops significantly as n increases. For $n = 1$ and $n = 2$, the fin efficiency is $<10\,\%$.

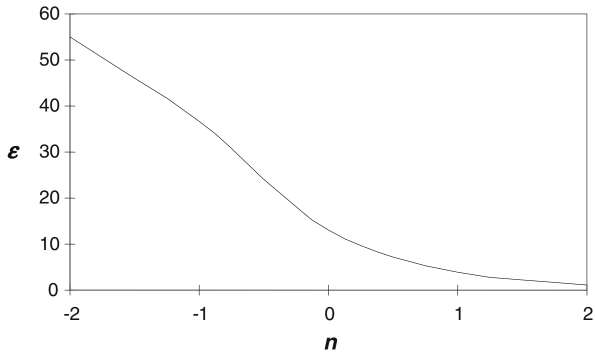

Fig. 7 Effect of exponent n on fin effectiveness with $a = 10\,\mathrm{W} \cdot \mathrm{m}^{-(n+1)} \cdot \mathrm{K}^{-1}$

Table 2 Average thermal conductivities in $W \cdot m^{-1} \cdot K^{-1}$ of the fin

$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$
500	69.31	10	1.5	0.23

Obviously, such functional grading of the fin material would not serve any useful purpose.

The effect of the exponent n on the fin effectiveness is illustrated in Fig. 7. Like the fin efficiency, the fin effectiveness also decreases as n increases and is <10 for $n=1$ and $n=2$ which as noted earlier would result in poor fin design.

6 Average Thermal-Conductivity Model

The spatially average thermal conductivity for the fin may be derived by integrating Eq. 1 with respect to r from the base of the fin to the tip of the fin which gives

$$\bar{k} = aF(r_\mathrm{b}, r_\mathrm{t}, n) = a\frac{1}{r_\mathrm{t}-r_\mathrm{b}}\frac{1}{n+1}\left(r_\mathrm{t}^{n+1} - r_\mathrm{b}^{n+1}\right), \quad n \neq -1, \tag{18}$$

$$\bar{k} = aF(r_\mathrm{b}, r_\mathrm{t}, n) = a\frac{1}{r_\mathrm{t}-r_\mathrm{b}}\ln(r_\mathrm{t}/r_\mathrm{b}), \quad n = -1. \tag{19}$$

For the specific fin under consideration, the average values of the thermal conductivity calculated from Eqs. 18 and 19 are provided in Table 2.

One may now define a new fin parameter $\bar{m}$ as

$$\bar{m} = \frac{m}{\sqrt{F(r_\mathrm{b}, r_\mathrm{t}, n)}} \tag{20}$$

and use it in Eqs. 16, 17, 5, and 6 to calculate the tip temperature, the heat transfer rate, the fin efficiency, and the fin effectiveness for the variable thermal conductivity of the fin. The results from the average thermal-conductivity model and the actual results are compared in Table 3 for the specific fin under consideration. The values based on the average thermal conductivity model are given in parentheses. For the tip temperature, the average thermal conductivity model over predicts the actual value by about 36 %

Table 3 Tip temperature, heat transfer rate, fin efficiency, and fin effectiveness: comparison of actual and average thermal-conductivity model values

n	-2	-1	0	1	2
$\bar{m}$	4.90	13.16	34.64	89.44	228.41
θ_t	87.80 (86.48)	43.46 (43.89)	4.84	0.014 (0.019)	0(0)
q	519.7 (508.7)	345.3 (324.6)	123.3	36.76 (44.46)	11.22 (16.87)
η	0.919 (0.899)	0.6107 (0.5737)	0.2180	0.065 (0.0786)	0.0200 (0.0298)
ε	55.13 (53.97)	36.63 (34.43)	13.08	3.90 (4.72)	1.180 (1.789)

for $n = -1$. In so far as the heat transfer rate is concerned, the values predicted by the average thermal conductivity model are lower than the exact values. The error between the two values is 2 % for $n = -2$ and 6 % for $n = -1$. For $n = 1$ and $n = 2$, the values predicted by the average thermal conductivity model are higher than the exact values. The error between the two values is 21 % for $n = -1$ and 50 % for $n = 2$. Regarding the fin efficiency, the average model under-predicts the exact values for $n = -2$ and $n = -1$ and over-predicts the exact values for $n = 1$ and $n = 2$. The maximum error between the two values is 49 % and occurs when $n = 2$. Finally, the results for the fin effectiveness indicate that the average model under-predicts the exact values for $n = -2$ and $n = -1$ but over-predicts the exact values for $n = 1$ and $n = 2$. The maximum error between the two results is 52 % and occurs when $n = 2$.

7 Conclusions

Some exact analytical solutions are reported for heat transfer in an annular fin of rectangular profile with the thermal conductivity varying with the radial coordinate. These solutions appear in terms of Airy functions, modified Bessel functions, hyperbolic functions, or power functions depending on the form of variation chosen. A spatially averaged thermal-conductivity model has also been described. The average model can introduce significant error in predicting the fin-tip temperature, the fin heat transfer rate, the fin efficiency, and the fin effectiveness. The error can be 2 % on the low side and as much as 52 % on the high side. These figures pertain to a specific fin situation studied in this paper and may vary from situation to situation. It is recommended that the exact solutions presented in the paper should be used and not the average model solutions, because the latter can introduce large errors under certain circumstances. The best fin design is achieved when the thermal conductivity of the fin varies inversely with the square of the radius. Although the values of the exponent $n < -2$ were not investigated, it is anticipated that the lower the value of n, the better is the performance of the fin. Future effort would be directed to determine if an optimum value of n exists that maximizes the fin performance.

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